Linear Algebra II

12/04/2010, Monday, 9:00-12:00

1 Gram-Schmidt process

Consider the vector space of P_3 with the inner product

$$\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$

(a) Is the basis $\{1, x, x^2\}$ an orthonormal basis? → $1, x^2$ nicht orthogonaal

(b) By applying the Gram-Schmidt process, find an orthonormal basis.

$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}x, \sqrt{\frac{3}{2}}(x^2-\frac{2}{3}$

(c) Find coördinates of the polynomial $1 + x + x^2$ in the orthonormal basis obtained above.

$\frac{5}{\sqrt{3}}, \sqrt{2}, \frac{2}{3}\sqrt{\frac{3}{2}}$

2 Singular value decomposition

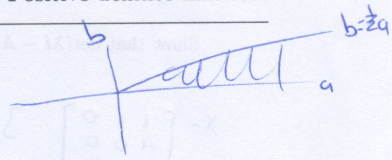
(a) Let $A \in \mathbb{R}^{n\times n}$. Show that A^TA and AA^T are similar. $S = A^T$

(b) Two matrices $A, B \in \mathbb{R}^{n\times n}$ are called unitarily equivalent if there exists an orthogonal matrix $W \in \mathbb{R}^{m\times m}$ such that $A = WBW^T$. Prove of disprove the statements:

(i) If two matrices are unitarily equivalent then they have the same singular values. ×

(ii) If two matrices have the same singular values then they are unitarily equivalent. ?

3 Positive definite matrices



(a) Let

$$A = \begin{bmatrix} a & b & 0 \\ b & b & b \\ 0 & b & a \end{bmatrix}$$

where a and b are real numbers. Plot the region of (a, b)-plane in which A is positive definite.

(b) Consider the function

$$f(x, y) = \frac{x}{y^2} + \frac{y}{x^2} + xy + 1.$$

(i) Show that $(1, 1)$ is a stationary point.

(ii) Determine the nature (local minimum, maximum, or saddle) of this stationary point.

minimum $(\lambda=3, \lambda=9)$

4 Cayley-Hamilton theorem

(a) Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix and $p_A(\lambda)$ be its characteristic polynomial. Define

$$q(\lambda) = \frac{1}{p_A(0)}\lambda^n p_A(\frac{1}{\lambda}).$$

$\frac{1}{A^{-1}} = A \quad p_A(A) = 0$

Show that $q(A^{-1}) = 0$.

(b) Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -3 \end{bmatrix}.$$

$(A+I)^{3k} = (-I)^k$ detor

Find $(A+I)^{3000}$.

I

$(-1)^{1000} I^{1000} = I$

5 Diagonalization and Jordan form

(a) Consider the matrix

$$\begin{bmatrix} a & b \\ 1 & a \end{bmatrix}$$

where a and b are real numbers. For which values of (a, b) is this matrix diagonalizable? alle. $b \neq 0$

(b) Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Show that $\det(\lambda I - A) = (\lambda - 1)^3$. Put it into the Jordan canonical form.

$$X = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad X^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$